Gravitational radiations of generic isolated horizons and non-rotating dynamical horizons from asymptotic expansions

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Instead of using a three dimensional analysis on quasi-local horizons, we adopt a four dimensional asymptotic expansion analysis to study the next order contributions from the nonlinearity of general relativity. From the similarity between null infinity and horizons, the proper reference frames are chosen from the compatible constant spinors for an observer to measure the energy-momentum and flux near quasi-local horizons. In particular, we focus on the similarity of Bondi-Sachs gravitational radiation for the quasi-local horizons and compare our results to Ashtekar-Kirshnan flux formular. The quasi-local energy momentum and flux of generic isolated horizons and non-rotating dynamical horizons are discussed in this paper.

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I. INTRODUCTION

The boundary of a black hole is defined as *a region of no escape of null rays*, which leads to an event horizon definition. However, the definition of the event horizon cannot give a realistic description of how a black hole grows since it is too global. The event horizon can be located only after an observer knows the global structure of space-time. Hence, the generalization of the event horizon to a *quasi-local* definition of horizons provides *an observer a possibility to detect horizons*. Ashtekar *et al* [1] proposed a notion of isolated horizons, which is quasi-locally defined, to describe equilibrium states of black holes. Unlike Killing horizons or stationary event horizons, they do not require any space-time Killing fields or exclude radiations outside horizons. If gravitational collapse occurs, the final stage of black holes is isolated and in the equilibrium state therefore will not radiate any more. However, there should still have gravitational or matter field radiations outside black holes. The definitions of generic isolated horizons yield a theoretical framework to describe these situations.

In order to clarify the idea of 'isolation', Ashtekar *et al* [1] started from the most general definition of isolated horizons called *non-expanding horizons* (NEHs). It requires the intrinsic (degenerate) metric q_{ab} on horizons to be time independent, i.e., $\mathcal{L}_\ell q_{ab} \hat{=} 0$, where $\hat{=}$ represents equal on horizon. Hence, ℓ, which is a null normal to horizon, can be considered as a Killing vector field of intrinsic horizon geometry. If one further requires the extrinsic curvature (the rotation 1-form) to be time independent, it then leads to the definition of *weakly isolated horizons* (WIHs). In WIHs, the black hole zeroth and first laws holds. By using a freedom of rescaling ℓ, it has been verified that there exists a specific ℓ which can reduce NEHs to WIHs. The most restricted generic isolated horizons are called *isolated horizons* (IHs), which require the full derivative operator $\mathcal{D}$ induced by space-time connection ∇ to be time independent.

It is expected that black holes are rarely in equilibrium in Nature. By using generic isolated horizons as a basis, the ideas can be generalized to *dynamical horizons* (DHs) definition by considering a space-like hypersurface rather than a null hypersurface in the NEH definition. The horizon geometry of dynamical horizon is time dependent and it allows a quantitative relation between the growth of the horizon area and the flux of energy and angular momentum across it [2]. Ref [2] adopted a 2+1 decomposition on a three dimensional space-like surface and the Cauchy data on the DHs must satisfy the scalar and vector constraints. Moreover, the relations between changes of the horizon area and energy fluxes cross the DHs were obtained and these fluxes has also been proved to be positive. However, this kind of approach does not tell us what is the gravitational free data near horizon when considering the full four dimensional space-time.

In contrast to Ashtekar *et al*'s [1, 2] three dimensional analysis, our works use 4-dimensional asymptotic expansions to study the neighborhoods of generic IHs and DHs. Since asymptotic expansion has been used to study gravitational radiations near the null infinity [8], it offers a useful scheme to analyze gravitational radiations approaching another boundary of space-time, horizons. We first set up a null frame with certain gauge choices near quasi-local horizons and then expand Newman-Penrose (NP) coefficients, Weyl, and Ricci curvature with respect to radius. Their fall-off can be deter-

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mined from NP equations, Bianchi equations, and exact solutions, e.g., the Kerr solution (see Appendix A) and the Vaidya solution (see Appendix B). This approach allows one to see the next order contributions from the nonlinearity of the full theory for the quasi-local horizons. The asymptotic expansions for the null infinity and the horizon are quite different geometrically. As we approach to null infinity, we consider an asymptotically flat space-time, however, the approach near horizons is *not necessarily asymptotically flat*. For the null infinity, we take the incoming null vector n^a as a generator of null infinity to generate different cuts with respect to different times, say u, and the outgoing null vector ℓ^a is parameterized by using affine parameter r. Moreover, ℓ_a is chosen as the gradient of the surface of a constant time u. For generic IHs, we take the outgoing null vector ℓ^a as the generator of horizons to generate different cross sections with respect to different times, say v. The ingoing null vector n^a is chosen as a tangent vector of ingoing null rays and also being parameterized by affine parameter r. The n_a is the gradient of the surface of a constant time v.

The main purpose of this paper is to calculate the amount of mass-energy fluxes crossing or near generic IHs and non-rotating DHs. Although there is no well-defined gravitational energy density in general relativity (GR), it does have well-defined mass or energy associated to a closed two-surface, i.e., quasi-locally. Without using Hamiltonian or 2+1 decomposition on these quasi-local horizons, we can use a quasi-local energy formula based on spinor fields to define the mass of a black hole. How can one measure the quasi-local energy-momentum when she or he is near black holes? Our strategy is to use the Nester-Witten 2-form together with similar concept of Bramson's frame alignment [4]. The Nester-Witten 2-form is commonly used to prove the positivity of energy and can be defined quasi-locally. It has been proved to be positive either for the stationary space-time or the space-time with radiation, i.e, the ADM mass or Bondi mass. Therefore we use it for a near horizon region which can allow radiation. However, what is *a good reference frame* for a strong gravitating field such as a black hole? To tackle this problem, we adopt the notion of frame alignment to find asymptotically constant spinors, which was done by Bramson for null infinity, and apply this framework on horizons. After obtaining the compatible constant spinors, we use them to define the quasi-local energy-momentum and flux near horizons. Since we know the changes of the constant spinors with respect to time, it is possible for us to see how does the energy flux cross horizons.

The plan of this paper is as follows. Section II gives a review of generic IHs and DHs. Their definitions and properties are discussed in terms of the NP coefficients. In Section III, we present a detail calculation of asymptotic expansions for IHs and non-rotating DHs. The asymptotic expansion provides a useful scheme to analyze the space-time geometry not only on quasi-lcoal horizons but also their neighborhoods. In Section IV, we apply the Bramson's frame alignment to IHs and non-rotating DHs and then find the compatible constant spinors. Thus, the good reference frame can be constructed from these constant spinors. On each cross section of quasi-local horizons, we find that constant spinors will satisfy the *Dogan-Mason's holomorphic condition*. In Section V, we obtain the quasi-local energy and flux of the generic IHs and non-rotating DHs by using these constant spinors. Although our approaches, which is based on gauge fixing quasi-local energy-momentum and *a time related condition*, are completely different from Ashtekar *et al*'s, it turns out to yield the same result as Ashtekar-Kirshnan gravitational fluxes for non-rotating DHs.

In this paper, we adopt the same notation as in the references [1, 2] for describing generic IHs and DHs. However, we choose the different convention $(+---)$, which is a standard convention for the NP formalism [7] (also see [5]). The Weyl tensor is completely specified by the five complex scalars $\Psi_0, ..., \Psi_4$ and the ten components of the Ricci tensor are defined in terms of the four real and three complex scalars $\Phi_{00}, ..., \Phi_{22}$. Their definitions can be found in p. 43-p. 44 of [5]. The necessary equations, i.e., commutation relations, NP equations and Bianchi identities, for asymptotical expansion analysis can be found in p. 45-p. 51 of [5].

II. DEFINITIONS OF QUASI-LOCAL HORIZONS

In this section, we give a review of generic IHs and DHs proposed by Ashtekar *et al* [1, 2]. The definitions of the quasi-local horizons yields several conditions on NP coefficients and these conditions are called gauge conditions throughout this paper.

A. The generic isolated horizons

We start from a 4-dimensional space-time manifold $(\mathscr{M}, g)$ with a 3-dimensional, null sub-manifold (Δ, q). The definition of non-expanding horizon is given as follows [1]:

a. Definition. Δ is called a *non-expanding horizon* (NEH) if (1) Δ is diffeomorphic to the product $S \times \mathbb{R}$ where S is a space-like 2-sphere. (2) The expansion $\Theta_{(\ell)}$ of any null normal ℓ to Δ vanishes, where the expansion is defined by $\Theta_{(\ell)} = \frac{1}{2} q^{ab} \nabla_a \ell_b$ with q_{ab} the degenerate intrinsic metric on Δ. (3) Einstein field equations hold on Δ and the stress-energy tensor T_{ab} is such that $T^a_b \ell^b$ is causal and future-directed on Δ.

The geometry of Δ is characterized by the intrinsic metric q and the induced derivative $\mathcal{D}$, i.e., $\mathcal{D}_a = \underleftarrow{\nabla_a}$, on $\triangle$.[1] The intrinsic degenerate metric q_{ab} on $\triangle$ has signature $(0, -, -)$. The vectors $(\ell^a, m^a, \overline{m}^a)$ span the tangent space to Δ with the dual co-frame given by the pull backs of $(n_a, m_a, \overline{m}_a)$. The expansions of outgoing and incoming null rays are defined by $\Theta_{(\ell)} := \frac{1}{2} q^{ab} \nabla_a \ell_b = -\mathrm{Re}[\rho]$, and

[1] The under-arrow $\leftarrow$ indicates the pullback of the index to $\triangle$.

$\Theta_{(n)} := \frac{1}{2}q^{ab}\nabla_a n_b = \mathrm{Re}[\mu]$, where $q^{ab} := -m^a\overline{m}^b - \overline{m}^a m^b$ on the tangent space of Δ. The twist and shear on Δ are defined as $\omega^2_{twist} := \frac{1}{2}q^a{}_c\ q^b{}_d\ \nabla_{[a}\ell_{b]}\nabla^{[c}\ell^{d]}$ and $|\sigma_{shear}| := [\frac{1}{2}q^a{}_c\ q^b{}_d\nabla_{(a}\ell_{b)}\nabla^{(c}\ell^{d)} - \Theta^2_{(\ell)}]^{\frac{1}{2}}$, respectively. Since ℓ is the null normal of the null hypersurface, it implies the twist free. Moreover, the shear vanishes by using Raychaudhuri equation [2] and the dominate energy condition. Therefore, the gauge conditions on NEH are

$$\kappa\hat{=}0, \sigma\hat{=}0, \rho\hat{=}0. \quad (1)$$

From (1), there must exist *a natural connection one-form* $\omega := \omega_a dx^a$ on $\triangle$ which can be obtained by

$$\mathcal{D}_a\ell^b\hat{=}\omega_a\ell^b. \quad (2)$$

The *surface gravity* $\kappa_{(\ell)}$ is defined as

$$\kappa_{(\ell)} := \omega_a\ell^a \quad (3)$$

on NEH Δ (measured by ℓ). Note that we do not have a unique normalization for ℓ. Under the scale transformation $\ell \mapsto f\ell$, we have $\omega \mapsto \omega + d\ln f$ and $\kappa_{(\ell)} \mapsto f\kappa_{(\ell)} + f\mathcal{L}_\ell \ln f$ which leaves Eq. (2) and Eq. (3) invariant.

From (2), we get

$$\mathcal{L}_\ell q_{ab}\hat{=}\underleftarrow{\mathcal{L}_\ell g_{ab}}\hat{=}q_{cb}\omega_a\ell^c + q_{ac}\omega_b\ell^c = 0 \quad (4)$$

for any null normal ℓ to $\triangle$. In fact, ℓ is an asymptotic Killing vector field as we approach the horizon even though *the space-time metric g_{ab} may not admit a Killing vector field in the neighborhood of* $\triangle$.

The energy condition in the third point of NEH definition then further implies that $R_{ab}\ell^b$ is proportional to ℓ_a [2], that is $R_{ab}\ell^a X^b\hat{=}0$, for any vector field X tangent to $\triangle$. We then have

$$\Phi_{00}\hat{=}\Phi_{01}\hat{=}\Phi_{10}\hat{=}0. \quad (5)$$

Since ℓ is expansion and shear-free, it must lie along one of the principal null directions of the Weyl tensor. From equation (b) and (k) in p. 46 of [5], we have

$$\Psi_0\hat{=}\Psi_1\hat{=}0, \quad (6)$$

so Ψ_2 is gauge invariant, i.e., independent of the choice of the null-tetrad vectors $(n, m, \overline{m})$, on $\triangle$. For NEH, we also have

$$d\omega\hat{=}2(\mathrm{Im}[\Psi_2])\ {}^2\epsilon \quad (7)$$

where ${}^2\epsilon$ is an area 2-form. The 2-form $d\omega$ can also be written as

$$2\mathcal{D}_{[a}\omega_{b]} = 2(\eth\pi - \bar{\eth}\overline{\pi})m_{[a}m_{b]}. \quad (8)$$

[2] For the outgoing null geodesic ℓ, the *Raychaudhuri equation* yields
$\mathcal{L}_\ell\Theta_{(\ell)} = -\Theta^2_{(\ell)} - \sigma_{shear}\overline{\sigma}_{shear} + \omega^2_{twist} + \kappa_{(\ell)}\Theta_{(\ell)} - \Phi_{00}$.

$\mathrm{Im}[\Psi_2]$ plays a role of gravitational contributions to the angular-momentum at $\triangle$. Ashtekar *et al* calls ω the *rotational 1-form potential* and $\mathrm{Im}[\Psi_2]$ the *rotational curvature scalar*. Therefore, π vanishes for a non-rotating $\triangle$.

Using the Cartan identity $\mathcal{L}_v = di_v + i_v d$ and (7), the Lie derivative of ω with respect to ℓ gives

$$\mathcal{L}_\ell\omega_a\hat{=}2\mathrm{Im}(\Psi_2)\,\ell^b\,{}^2\epsilon_{ba} + \mathcal{D}_a(\ell^b\omega_b)\,\hat{=}\mathcal{D}_a\kappa_{(\ell)}. \quad (9)$$

On NEH, the surface gravity may not be constant. To obtain the black hole zeroth law such that the surface gravity is constant, one may need a further condition, i.e., $\mathcal{L}_\ell\omega_a = 0$, on NEH. It motivates the definition of *weakly isolated horizon*.

b. Definition. A *weakly isolated horizon* (WIH) is a NEH with an equivalence class of null normals under constant transformation. The flow of ℓ preserves the rotation 1-form ω $\mathcal{L}_\ell\omega_a\hat{=}0$,i.e., $[\mathcal{L}_\ell, \mathcal{D}]\ell\hat{=}0$.

From (9), the condition of WIH basically preserves the black hole zeroth law. Because ℓ is tangent to $\triangle$, the evolution equation is in fact a constraint. See (B21) and (B22) of [1]. Therefore, given a NEH, we can select a canonical $[\ell]$ by requiring $(\triangle, [\ell])$ to be a WIH satisfying

$$\mathcal{L}_\ell\mu\hat{=}0 \quad \text{or} \quad \dot{\mu}\hat{=}0. \quad (10)$$

$\triangle$ generically admits a unique $[\ell]$ such that the incoming expansion is time independent. This result will establish that a generic NEH admits a unique $[\ell]$ such that $(\triangle, [\ell])$ is a WIH on which the incoming expansion μ is time independent.

c. Definition. A weakly isolated horizon $(\Delta, [\ell])$ is said to be *isolated horizon* (IH) if $[\mathcal{L}_\ell, \mathcal{D}]V\hat{=}0$, for all vector fields V tangential to Δ and all $\ell \in [\ell]$.

From this definition, we have $[\mathcal{L}_\ell, \mathcal{D}]\ell\hat{=}0$ and $[\mathcal{L}_\ell, \mathcal{D}]n\hat{=}0$. The first one gives the surface gravity to be a constant by previous argument. So $\dot{\epsilon}\hat{=}0$. The second one gives $\dot{\pi}\hat{=}\dot{\mu}\hat{=}\dot{\lambda}\hat{=}0$.

B. The dynamical horizon

The generic IHs are taken as the equilibrium state of the DHs. The DHs can be foliated by marginally trapped surface S. Therefore, the expansion of the outgoing tetrad vanishes.

d. Definition A smooth, 3 dimensional, space-like submanifold H of space-time is said to be a *dynamical horizon* (DH) if it can be foliated by a family of closed 2-manifold such that: (1) on each leaf, S, the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes, (2) the expansion $\theta_{(n)}$ of the other null normal n^a is negative.

Here, $\theta_{(\ell)} := \frac{1}{2}\,{}^{(2)}q^{ab}\nabla_a\ell_b$ and $\theta_{(n)} := \frac{1}{2}\,{}^{(2)}q^{ab}\nabla_a n_b$ where ${}^{(2)}q^{ab} = -(m^a\overline{m}^b + m^b\overline{m}^a)$ is intrinsic to the cross section S of H. From this definition, it basically tells us a dynamical horizon is a space-like hypersurface which is foliated by closed, marginally trapped two surface. The requirement of

the expansion of the incoming null normal is strictly negative since we want to study a black hole (future horizon) rather than a white hole. Also, it implies

$$\mathrm{Re}[\rho]\hat{=}0. \tag{11}$$

The angular momentum associated with cross-section S is $J_S^\psi = -\frac{1}{8\pi}\oint_S K_{ab}\psi^a R^b dS$, where ψ^a is the rotational vector field and need not be an axial Killing field and K_{ab} is the extrinsic three curvature [2]. ψ^a can be written as linear combinations of m^a and $\overline{m}^a$, i.e., $\psi^a = Em^a + \overline{E}\overline{m}^a$ where $E \neq 0$. The extrinsic three curvature K_{ab} can be decomposed into $A^{(2)}q_{ab} + S_{ab} + 2W_{(a}R_{b)} + BR_aR_b$ which has been expressed in terms of Newman-Penrose coefficients [9, 10]. If we choose ℓ to be geodesic, $n_a = \nabla_a v$ and $m, \overline{m}$ tangent to the two surface [9, 10], we get

$$J_S^\psi = \frac{1}{8\pi}\oint_S \mathrm{Re}[\pi\overline{E}]dS. \tag{12}$$

Hence, if $\pi\hat{=}0$, it implies a non-rotating DH.

If we use the gauge conditions $\kappa\hat{=}\pi - \overline{\tau}\hat{=}\pi - (\alpha + \overline{\beta})\hat{=}0$ on a rotating DH, the total flux of Ashtekar-Krishnan [2] [10] then becomes

$$F_{\text{total}} = \frac{1}{4\pi}\int [|\sigma|^2 + |\pi|^2 + \Phi_{00}]Nd^3V \tag{13}$$

where the gravitational flux is

$$F_{grav} = \frac{1}{4\pi}\int_{\Delta H} N(|\sigma|^2 + |\pi|^2)d^3V. \tag{14}$$

and the matter flux of Vaidya solution is

$$F_{\text{matter}} :\ = \int_H T_{ab}T^a\ell^b Nd^3V = \frac{1}{4\pi}\int \Phi_{00}Nd^3V \tag{15}$$

where we use $4\pi T_{ab}\ell^a\ell^b = \Phi_{00}$.

III. ASYMPTOTIC EXPANSIONS NEAR QUASI-LOCAL HORIZONS

A. Near generic isolated horizons: vacuum

Frame setting, gauge choice and gauge conditions

We choose the incoming null tetrad $n_a = \nabla_a v$ to be gradient of the null hypersurface $v = const.$ and it gives $g^{ab}v_{,a}v_{,a} = 0$. We further choose $m, \overline{m}$ tangent to the two surface. These gauge choices lead to

$$\begin{aligned} \nu &= \mu - \overline{\mu} = \rho - \overline{\rho} = \gamma + \overline{\gamma} = \pi - \alpha - \overline{\beta} = 0, \\ \pi &= \overline{\tau}. \end{aligned} \tag{16}$$

From the definition of NEH, the gauge conditions from eq (1) can be expanded with respect to $r' = r - r_\Delta$ as

$$\kappa = \kappa_0 r' + O(r'^2), \rho = \rho_0 r' + O(r'^2), \sigma = \sigma_0 r' + O(r'^2),$$

where $\rho_0 := [\Psi_2^0 - \overline{\eth}_0\overline{\pi}_0 + \pi_0\overline{\pi}_0]$, $\sigma_0 := [-\overline{\eth}_0\overline{\pi}_0 + \pi_0\overline{\pi}_0]$ from asymptotic expansions for NP equations **(q)** and **(p)** in p. 47 of [5] [3]. Further we choose ℓ to be flag planes parallel on NEH, therefore it leads to

$$\epsilon - \overline{\epsilon} = O(r'). \tag{17}$$

The rest of NP coefficients are $O(1)$. The Weyl tensor has the fall off (refer to equation (6))

$$\Psi_0 = O(r'), \Psi_1 = O(r'). \tag{18}$$

In order to preserve orthogonal relation $\ell^a n_a = 1, m^a\overline{m}_a = -1, \ell^a m_a = n^a m_a = 0$, we can choose the tetrad as

$$\ell^a = (1, U, X^3, X^4), \quad n^a = (0, -1, 0, 0), \quad m^a = (0, 0, \xi^3, \xi^4),$$

where U, X^3, X^4 are real and ξ^3, ξ^4 are complex.

We first expand NP spin coefficients, tetrad components U, X^k, ξ^k and Weyl spinors Ψ_k with respect to r' and substitute them into NP equations which include the commutation relation, Ricci identities equations and the Bianchi identities equations. By examining equation **(a)** in p. 46, it implies U must be $O(r')$. X^k will become $O(r')$ since we can use the condition (4.86) in [9] to perform the coordinate transformation to make $X^{0k} = 0$.

The radial NP equations

From **(n)**,**(j**,**(r)**,**(o)**,**(q)**,**(p)**,**(i)**,**(f)** in p. 46-47 :

$$\begin{aligned}
\mu &= \mu_0 + ({\mu_0}^2 + \lambda_0\overline{\lambda}_0)r' + O(r'^2), \\
\lambda &= \lambda_0 + (2\mu_0\lambda_0 + 4\lambda_0\gamma_0 + \Psi_4^0)r' + O(r'^2), \\
\alpha &= \alpha_0 + (\lambda_0(\overline{\pi}_0 + \beta_0) + 2\gamma_0\alpha_0 + \alpha_0\mu_0 + \Psi_3^0 - \overline{P}\overset{c}{\nabla}\gamma_0)r' \\
&\quad + O(r'^2), \\
\beta &= \beta_0 + (\mu_0\overline{\pi}_0 - \beta_0(2\gamma_0 - \mu_0) + \alpha_0\overline{\lambda}_0 - P\overset{c}{\nabla}\gamma_0)r' \\
&\quad + O(r'^2), \\
\rho &= [\Psi_2^0 - \overline{\eth}_0\overline{\pi}_0 + \pi_0\overline{\pi}_0]r' + O(r'^2), \\
\sigma &= (-P\overset{c}{\nabla}\overline{\pi}_0 + 2\beta_0\overline{\pi}_0)r' + O(r'^2), \\
\pi &= \pi_0 + [2(\mu_0 + \gamma_0)\pi_0 + 2\lambda_0\overline{\pi}_0 + \Psi_3^0]r' + O(r'^2), \\
\epsilon &= \epsilon_0 + [2\alpha_0\overline{\pi}_0 + 2\beta_0\pi_0 - 2\gamma_0\epsilon_0 + \overline{\pi}_0\pi_0 + \Psi_2^0 - \dot{\gamma}_0]r' \\
&\quad + O(r'^2) \\
\kappa &= \kappa_0 r' + O(r'^2)
\end{aligned}$$

From (305) and (303) in p. 45:

$$\begin{aligned} \xi^k &= \xi^{k0} + O(r'), \ \ U = 2\epsilon_0 r' + O(r'^2), \\ X^k &= 2(\pi_0\xi^{k0} + \overline{\pi}_0\overline{\xi}^{k0})r' + O(r'^2). \end{aligned} \tag{19}$$

From **(e)**, **(f)**,**(g)** and **(h)** in p. 46:

$$\begin{aligned}
\Psi_0 &= \tfrac{1}{2}(-\eth_0\Psi_1^0 + 4\overline{\pi}_0\Psi_1^0 - 3\sigma_0\Psi_2^0)r'^2 + O(r'^3), \\
\Psi_1 &= (-\eth_0\Psi_2^0 + 3\overline{\pi}_0\Psi_2^0)r' + O(r'^2), \\
\Psi_2 &= \Psi_2^0 + (-\eth_0\Psi_3^0 + 2\overline{\pi}_0\Psi_3^0 + 3\mu_0\Psi_2^0)r' + O(r'^2), \\
\Psi_3 &= \Psi_3^0 + (-\eth_0\Psi_4^0 + \overline{\pi}_0\Psi_4^0 + 4\mu_0\Psi_3^0)r' + O(r'^2),
\end{aligned}$$

[3] From now on, we only indicate the equation and page numbers without mention the ref [5].

The non-radial NP equations

From **(a)**,**(b)**,**(c)**, **(g)**,**(d)**,**(e)**,**(h)**,**(k)**,**(m)**,**(l)** in p.46-47:

$$\begin{array}{lll}
\dot{\rho}_0 & = \bar{\eth}_0\kappa_0 - \bar{\kappa}_0\bar{\pi}_0 - \kappa_0\pi_0, & \dot{\sigma}_0 = \eth_0\kappa_0, \\
\dot{\pi}_0 & + \ \kappa_0 = 0, & \\
\dot{\lambda}_0 & = \bar{\eth}_0\pi_0 - \pi_0\bar{\pi}_0 - 2\lambda_0\epsilon_0, & \\
\dot{\alpha}_0 & - \ \overline{P}\overset{\bar{c}}{\nabla}\epsilon_0 = 0, & \dot{\beta}_0 - P\overset{c}{\nabla}\epsilon_0 = 0, \\
\dot{\mu}_0 & = \eth_0\pi_0 + \pi_0\bar{\pi}_0 - 2\mu_0\epsilon_0 + \Psi_2^0, & \\
\eth_0\rho_0 & - \ \bar{\eth}_0\sigma_0 = -\Psi_1^0, & \eth_0\lambda_0 - \bar{\eth}_0\mu_0 = -\Psi_3^0, \\
\Psi_2^0 & = \overline{P}\overset{\bar{c}}{\nabla}\beta_0 - P\overset{c}{\nabla}\alpha_0 + \alpha_0\bar{\alpha}_0 & +\beta_0\overline{\beta}_0 - 2\alpha_0\beta_0, \\
 & 2\mathrm{Im}\eth_0\pi_0 = -2\mathrm{Im}\Psi_2^0. &
\end{array} \tag{20}$$

From (304) , (306) in p. 45:

$$\kappa_0 \ = \ -2P\overset{c}{\nabla}\epsilon_0, \ \ \dot{P} = 0, \ \ \overline{P}\overset{\bar{c}}{\nabla}\ln P = \beta_0 - \bar{\alpha}_0. \tag{21}$$

From **(a)**,**(b)**,**(c)**, **(d)** in p. 49:

$$\begin{array}{lll}
\dot{\Psi}_1^0 & = & 2\epsilon_0\Psi_1^0 - 3\kappa_0\Psi_2^0, \ \ \dot{\Psi}_2^0 = -2\kappa_0\Psi_3^0, \\
\dot{\Psi}_3^0 & - & \bar{\eth}_0\Psi_2^0 = 3\pi_0\Psi_2^0 - 2\epsilon_0\Psi_3^0 - \kappa_0\Psi_4^0, \\
\dot{\Psi}_4^0 & - & \bar{\eth}_0\Psi_3^0 = -3\lambda_0\Psi_2^0 + 4\pi_0\Psi_3^0 - 4\epsilon_0\Psi_4^0,
\end{array}$$

where the complex derivative is defined as $\overset{c}{\nabla} := \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}$, $P(v, x^k) := \xi^{30} = -i\xi^{40}$ and $P\overset{c}{\nabla} = \delta_0$.

From **(d)** in p. 46 and complex conjugate of **(e)** in p. 46 , we have

$$\dot{\pi}_0 = \frac{d}{dv}(\alpha_0 + \overline{\beta}_0) = 2\overline{P}\overset{c}{\nabla}\epsilon_0 = -\bar{\kappa}_0. \tag{22}$$

Using **(c)** in p. 46, $\dot{\pi}_0 = -\kappa_0$. It implies $\kappa_0 - \bar{\kappa}_0 = 0$. Therefore κ_0 is real. Using **(a)** in p. 46 and the fact that $\Psi_3^0 \neq 0$ for the rotating NEH , we get

$$\kappa_0 = 0, \text{i.e.,} \quad \delta_0\epsilon_0 = 0, \tag{23}$$

(i.e., $P\overset{c}{\nabla}\epsilon_0 = 0$) on rotating NEH.

Surface gravity: from NEH to WIH

Here we prove that the surface gravity for a rotating WIH is also constant. We make a coordinate choice $r_0 = -\frac{1}{\mu_0}$ on the NEH and it becomes a WIH. This gives $\dot{\mu}_0 = 0$. Applying time derivative on **(h)** in p. 46 and using $\kappa_0 = 0$ from **(b)** in p. 49, we then get $\dot{\epsilon}_0 = 0$.

It then gives us that ϵ_0 is constant on WIH. So the surface gravity $\kappa_{(\ell)} \hat{=} \mathrm{Re}\epsilon_0$ is constant on WIH. For NEH, the surface gravity is not necessary constant.

B. Near a non-rotating dynamical horizon: non-vacuum expansion

As before, we use advanced Eddington-Finkelstein type coordinates (v, r, θ, ϕ). To start with, we choose $n_a = \partial_a v$ along null hypersurface $v = const$. We have $g^{ab}v_{,a}v_{,a} = 0$. This gives the gauge conditions $\nu = \mu - \overline{\mu} = \gamma + \overline{\gamma} = \overline{\alpha} + \beta - \overline{\pi} = 0$. Further we can choose n^a flag plane parallel, this implies $\gamma = 0$. The incoming tetrad takes the form $n^a = -\frac{dx^a}{dr} = g^{ab}v_{,b} = \delta_2^\mu = (0, -1, 0, 0)$. $m, \overline{m}$ are chosen to be tangent to the two surface. Hence, $\rho = \overline{\rho}, \pi = \overline{\tau}$. Applying the null rotation type III $\ell \to A^{-1}\ell, m \to e^{2i\theta}m$, we can chose θ to make $\mathrm{Im}[\epsilon] = 0$.

Gauge choices. We conclude that our total gauge choices of the frame in this section are

$$\nu = \rho - \overline{\rho} = \mu - \overline{\mu} = \gamma = \epsilon - \overline{\epsilon} = 0, \pi = \overline{\tau} = \alpha + \overline{\beta}. \tag{24}$$

In order to preserve orthonormal relation,

$$\ell^a n_a = 1, m^a\overline{m}_a = -1, \ell^a m_a = n^a m_a = 0,$$

we can choose the tetrad as

$$\ell^a = (1, U, X^3, X^4), \ n^a = (0, -1, 0, 0), \ m^a = (0, 0, \xi^3, \xi^4).$$

Transfer to new coordinates (v, r', θ, ϕ)

Since the dynamical horizon is not a null hypersurface, we have to transfer to a 'good' coordinates which can capture this picture. We defined a new radius coordinate by using

$$r' = r - r_\Delta(v).$$

To tackle the problem of the power series expansion near a non-null hypersurface, we make a coordinate transformation by using $dr' = dr - \dot{r}_\Delta dv$ to transfer coordinates into the new coordinates (v, r', θ, ϕ). The tetrad in the new coordinates (v, r', θ, ϕ) are:

$$\begin{array}{l}
\ell^a = (1, U - \dot{r}_\Delta, X^3, X^4), \\
n^a = (0, -1, 0, 0), \ m^a = (0, 0, \xi^3, \xi^4).
\end{array}$$

The metric has the form

$$g^{ab} = \ell^a n^b + n^a \ell^b - m^a \overline{m}^b - \overline{m}^a m^b,$$

where it's components are

$$\begin{array}{l}
g^{01} = -1, \ g^{00} = g^{0k} = 0, \ g^{11} = -2U + 2\dot{r}_\Delta, \\
g^{1k} = -X^k, \ \ g^{mn} = -(\xi^m\overline{\xi}^n + \overline{\xi}^m\xi^n).
\end{array}$$

H is a non-null hypersurface, since $g^{ab}r'_{,a}r'_{,b} = -2U + 2\dot{r}_\Delta$ in this new coordinates.

Gauge conditions of non-rotating dynamical horizons

On the non-rotating dynamical horizons, the null normal ℓ_a is hypersurface orthogonal, therefore it is geodesic and twist-free. Moreover, it is the null normal of the marginal trapping surface, therefore it is expansion-free on the dynamical horizon. However, unlike the isolated horizon, the shear is non-vanishing on the dynamical horizon. Here we only consider a non-rotating dynamical horizon, so π vanishes on the boundary. In terms of NP, these conditions are

$$\kappa \hat{=} \rho \hat{=} \pi \hat{=} 0$$

on the non-rotating DHs. Therefore, the fall off of NP coefficients and the components of the null tetrad are

$$\begin{aligned}&U, \xi^k, X^k, \mu, \epsilon, \alpha, \beta, \lambda, \sigma, = O(1),\\ &\kappa, \rho, \pi = O(r').\end{aligned}$$

The falloff of the Weyl tensor and stress energy tensor

Because we want to compare with the Vaidya solution, we assume that the falloff of the Weyl tensors and Ricci tensors is the same as that of Vaidya solution. Therefore, we take the Weyl tensor as

$$\Psi_0 = O(r'), \Psi_1 = O(r'), \Psi_2 = \Psi_3 = \Psi_4 = O(1) \tag{25}$$

and the Ricci tensors as

$$\Phi_{00} = O(1), \Phi_{22} = \Phi_{11} = \Phi_{02} = \Phi_{01} = \Phi_{21} = O(r'). \tag{26}$$

We note that the coefficient of order $O(1)$ of the tetrad components U, X^k, i.e., U_0, X^{0k} can be made to be vanished by using coordinate transformation.

The radial equations

From **(n)**,**(j**,**(r)**,**(o)**,**(q)**,**(p)**,**(i)**,**(f)**,**(c)** in p. 46-47 :

$$\begin{array}{rcl}\mu &=& \mu_0 + ({\mu_0}^2 + \lambda_0\overline{\lambda}_0)r' + O(r'^2),\\ \lambda &=& \lambda_0 + (2\mu_0\lambda_0 + \Psi_4^0)r' + O(r'^2),\\ \alpha &=& \alpha_0 + (\alpha_0\overline{\mu}_0 - \lambda_0\overline{\alpha}_0 + \Psi_3^0)r' + O(r'^2),\\ \beta &=& -\overline{\alpha}_0 + (\alpha_0\overline{\lambda}_0 - \mu_0\overline{\alpha}_0)r' + O(r'^2),\\ \rho &=& (\sigma_0\lambda_0 + \Psi_2^0)r' + O(r'^2),\\ \sigma &=& \sigma_0 + \mu_0\sigma_0 r' + O(r'^2),\\ \pi &=& \Psi_3^0 r' + (\mu_0\pi_0 + \lambda_0\overline{\pi}_0)r'^2 + O(r'^3),\\ \epsilon &=& \epsilon_0 + \Psi_2^0 r' + O(r'^2),\\ \kappa &=& \dot{r}_\Delta\overline{\Psi}_3^0 r' + O(r'^2),\end{array} \tag{27}$$

From (305) and (303):

$$\begin{array}{rcl}\xi^k &=& \xi^{k0} + (\overline{\lambda}_0\overline{\xi}^{k0} + \mu_0\xi^{k0})r' + O(r'^2),\\ U &=& 2\epsilon_0 r' + \Psi_2^0 r'^2 + O(r'^3),\\ X^k &=& (\Psi_3^0\xi^{k0} + \overline{\Psi}_3^0\xi^{k0})r'^2 + O(r'^3).\end{array} \tag{28}$$

From **(e)**, **(f)**,**(g)**, **(h)**, **(i)**,**(j)** and **(k)** in p. 49-51 :

$$\begin{array}{rcl}\Psi_0 &=& (-3\sigma_0\Psi_2^0 + \overline{\lambda}_0\Phi_{00}^0)r' + O(r'^2),\\ \Psi_1 &=& (-2\sigma_0\Psi_3^0 - \eth_0\Psi_2^0)r' + O(r'^2),\\ \Psi_2 &=& \Psi_2^0 + (-\eth_0\Psi_3^0 + 3\mu_0\Psi_2^0 - \sigma_0\Psi_4^0)r' + O(r'^2),\\ \Psi_3 &=& \Psi_3^0 + (-\eth_0\Psi_4^0 + 4\mu_0\Psi_3^0)r' + O(r'^2),\\ \Psi_4 &=& \Psi_4^0 + \Psi_4^1 r' + O(r'^2),\\ \Phi_{00} &=& \Phi_{00}^0 + 2\mu_0\Phi_{00}^0 r' + O(r'^2),\\ \Phi_{11} &=& \frac{1}{2}\dot{r}_\Delta\Phi_{22}^0 r'^2 + O(r'^3),\\ \Phi_{01} &=& -\dot{r}_\Delta\Phi_{12}^0 r'^2 + O(r'^3),\\ \Phi_{22} &=& \Phi_{22}^0 r'^2 + O(r'^3), \Phi_{12} = \Phi_{12}^0 r'^2 + O(r'^3),\\ \Phi_{02} &=& \Phi_{02}^0 r'^2 + O(r'^2).\end{array} \tag{29}$$

The non-radial NP equations

From (304) and (306) in p. 45:

$$P \overset{c}{\nabla} \dot{r}_\Delta = 0, \;\; \dot{P} = \dot{r}_\Delta(\overline{\lambda}_0\overline{P} + \mu_0 P), \;\; \alpha_0 = \tfrac{1}{2}\overline{P} \overset{\bar{c}}{\nabla} \ln P. \tag{30}$$

From **(a)**,**(b)**,**(c)**, **(g)**,**(d)**,**(e)**,**(m)**,**(l)**,**(n)** in p. 46-47 :

$$\begin{aligned}&\dot{r}_\Delta(\Psi_2^0 + \sigma_0\lambda_0) = -\Phi_{00}^0 - \sigma_0\overline{\sigma}_0,\\ &\dot{\sigma}_0 = 2\epsilon_0\sigma_0 + \dot{r}_\Delta\mu_0\sigma_0,\\ &\kappa_0 = -2P \overset{c}{\nabla} \epsilon_0 = \dot{r}_\Delta\overline{\Psi}_3^0,\\ &\dot{\lambda}_0 = -2\epsilon_0\lambda_0 + \mu_0\overline{\sigma}_0 + \dot{r}_\Delta(2\mu_0\lambda_0 + \Psi_4^0),\\ &\dot{\alpha}_0 = \dot{r}_\Delta(\mu_0\alpha_0 - \lambda_0\overline{\alpha}_0 + \tfrac{1}{2}\Psi_3^0) - \overline{\alpha}_0\overline{\sigma}_0,\\ &\Psi_3^0 = \overline{\eth}_0\mu_0 - \eth_0\lambda_0,\\ &\Psi_2^0 = \tfrac{P\overline{P}}{2}(2 \overset{c}{\nabla} \ln\overline{P} \overset{\bar{c}}{\nabla} \ln P - \overset{c}{\nabla} \ln\overline{P} - \overset{\bar{c}}{\nabla} \ln P) - \lambda_0\sigma_0,\\ &\mathrm{Im}\Psi_2^0 = -\mathrm{Im}(\lambda_0\sigma_0),\\ &\Psi_2^0 = \partial_v\mu_0 + 2\epsilon_0\mu_0 - \lambda_0\sigma_0 - \dot{r}_\Delta(\mu_0^2 + \lambda_0\overline{\lambda}_0).\end{aligned}$$

From **(a)**,**(b)**,**(c)**, **(d)** in p. 49:

$$\begin{aligned}&\dot{r}_\Delta(2\sigma_0\Psi_3^0 + \eth_0\Psi_2^0) = -P \overset{c}{\nabla} \Phi_{00}^0,\\ &\partial_v\Psi_1^0 = \overline{P} \overset{\bar{c}}{\nabla} \Psi_0^0 + 2\epsilon_0\Psi_1^0 - 3\kappa_0\Psi_2^0 + \overline{\pi}_0\Phi_{00}^0\\ &\qquad -2\dot{r}_\Delta(\Phi_{01}^0 - \Psi_1^1),\\ &\partial_v\Psi_2^0 = \mu_0\Phi_{00}^0 + \dot{r}_\Delta(-\eth_0\Psi_3^0 + 3\mu_0\Psi_2^0 - \sigma_0\Psi_4^0),\\ &\partial_v\Psi_3^0 = \overline{\eth}_0\Psi_2^0 - 2\epsilon_0\Psi_3^0 + \dot{r}_\Delta(-\eth_0\Psi_4^0 + 4\mu_0\Psi_3^0),\\ &\partial_v\Psi_4^0 = \overline{\eth}_0\Psi_3^0 - 3\lambda_0\Psi_2^0 - 4\epsilon_0\Psi_4^0 + \dot{r}_\Delta\Psi_4^1.\end{aligned} \tag{31}$$

For example, $\dot{\sigma}_0$ has a next order contribution from asymptotic expansion since $\dot{\sigma}_0 = 2\epsilon_0\sigma_0 + \dot{r}_\Delta\sigma_1$, where $\sigma_1 = \mu\sigma_0$.

IV. CONSTANT SPINORS FOR QUASI-LOCAL HORIZONS

A. Constant spinors for the generic isolated horizons: Frame alignment

In this section, we adopt a similar idea of Bramson's asymptotic frame alignment for null infinity [4] and apply it to set up spinor frames on the quasi-local horizons. We define the spinor frames

$$Z_A{}^{\underline{A}} = (\lambda_A, \mu_A) \tag{32}$$

where $\lambda_A = \lambda_1 o_A - \lambda_0\iota_A$, $\mu_A = \mu_1 o_A - \mu_0\iota_A$. We expand λ_1, λ_0 as

$$\lambda_1 = \lambda_1^0(v,\theta,\phi) + \lambda_1^1(v,\theta,\phi)r' + O(r'^2), \tag{33}$$
$$\lambda_0 = \lambda_0^0(v,\theta,\phi) + \lambda_0^1(v,\theta,\phi)r' + O(r'^2). \tag{34}$$

Here λ_1 is type $(-1,0)$ and λ_0 is type $(1,0)$.

Firstly, we demand the conditions on spin frames to be parallelly transported along null normal ℓ^a direction on the quasi-local horizons, so

$$\mathrm{lim}_{r'\to 0}\, DZ_A{}^{\underline{A}} = 0, \tag{35}$$

and also the conditions of the frames on different generators on the quasi-local horizons are:

$$\mathrm{lim}_{r'\to 0}\, \delta Z_A{}^{\underline{A}} = \mathrm{lim}_{r'\to 0}\, \overline{\delta} Z_A{}^{\underline{A}} = 0. \tag{36}$$

We then get six conditions

$$
\begin{aligned}
\dot{\lambda}_0^0 - \epsilon_0\lambda_0^0 &= 0, \text{i.e.,} \quad \text{þ}_0\lambda_0^0 = 0, && (37)\\
\dot{\lambda}_1^0 + \epsilon_0\lambda_1^0 &= \pi_0\lambda_0^0, \text{i.e.,} \quad \text{þ}_0\lambda_1^0 = \pi_0\lambda_0^0, && (38)\\
\eth_0\lambda_0^0 &= 0, && (39)\\
\eth_0\lambda_1^0 - \mu_0\lambda_0^0 &= 0, && (40)\\
\eth_0'\lambda_0^0 &= 0, && (41)\\
\eth_0'\lambda_1^0 + \sigma_0'\lambda_0^0 &= 0. && (42)
\end{aligned}
$$

for the constant spinors λ_A on the generic isolated horizons. In (42) and also the following, we use the symbol $-\sigma_0'$ to represent the one of the NP coefficients, λ_0 (the leading order of NP shear of n), in order to avoid confusion with the spinor λ_A

We use the condition (37) and the fact that $\text{þ}_0\eth_0 = \eth_0\text{þ}_0$ on the horizon. Apply þ_0 on (39), we find

$$0 = \text{þ}_0\eth_0\lambda_0^0 = \eth_0\text{þ}_0\lambda_0^0 = 0. \quad (43)$$

So condition (37) and (39) are compatible.

Apply þ on (40) and use condition (37) and (38), we have

$$0 \; = \; \Psi_2^0\lambda_0^0. \quad (44)$$

Hence the condition (37), (38) and (40) are not compatible unless $\Psi_2^0 = 0$.

From the previous analysis, we conclude that *the compatible frame alignment conditions for the generic isolated horizon* are (37), (39) and (40). Equation (39) and (40) are Dougan-Mason's holomorphic conditions [6]. Here we see that the conditions of the spinor field to be asymptotically constant on NEH implies the Dougan-Mason holomorphic conditions on the cuts of the NEH. These equations will be used together with the Nester-Witten two form to define the quasi-local energy-momentum. The time related condition (37) will tell us how the energy momentum changes with time along NEH and will be useful to calculate the energy flux across the horizon.

B. Constant spinors for a non-rotating dynamical horizon

The dynamical horizon is a space-like or time-like hypersurface if the horizon is expanding $\dot{r}_\Delta > 0$ or contracting. From this we know that it's generator is non-null and it is a time-like or space-like generator respectively. To understand this, we have to work in the (v, r', θ, ϕ) coordinate. The dynamical horizon generator R is tangent to the dynamical horizon where $\mathcal{L}_R v = 1$ so $R = \frac{d}{dv}$. We have the relation

$$
\begin{gathered}
R^a = \ell^a - \dot{r}_\Delta n^a = \frac{\partial}{\partial v},\\
R_a = \ell_a - \dot{r}_\Delta n_a = -2\dot{r}_\Delta dv - dr', R^a R_a = -2\dot{r}_\Delta.
\end{gathered}
$$

So if $\dot{r}_\Delta > 0$ then the H is a space-like hypersurface. Since $g^{ab}r'_{,a}r'_{,b} = 2\dot{r}_\Delta$.

We define the spin frame $Z_A{}^{\underline{A}} = (\lambda_A, \mu_A)$ where

$$
\begin{aligned}
\lambda_A &= \lambda_1 o_A - \lambda_0 \iota_A, && (45)\\
\mu_A &= \mu_1 o_A - \mu_0 \iota_A, && (46)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_0 &= \lambda_0^0(v,\theta,\phi) + \lambda_0^1(v,\theta,\phi)r' + O(r'^2), && (47)\\
\lambda_1 &= \lambda_1^0(v,\theta,\phi) + \lambda_1^1(v,\theta,\phi)r' + O(r'^2), && (48)
\end{aligned}
$$

where λ_1 is type $(-1, 0)$ and λ_0 is type $(1, 0)$.

Firstly, we demand that the conditions on spin frames to be parallelly transported along the non-rotating dynamical horizon generators R^a direction on H, so

$$\lim_{r'\to 0} R Z_A{}^{\underline{A}} = \lim_{r'\to 0}(D - \dot{r}_\Delta \Delta) Z_A{}^{\underline{A}} = 0, \quad (49)$$

where we use $R^a = (\ell^a - \dot{r}_\Delta n^a)$. The conditions of the frames on different generators on H are:

$$\lim_{r'\to 0} \delta Z_A{}^{\underline{A}} = \lim_{r'\to 0} \; \overline{\delta} Z_A{}^{\underline{A}} = 0. \quad (50)$$

Therefore, from $o^A(D - \dot{r}_\Delta\Delta)\lambda_A$ in equation (49) we get

$$(D - \dot{r}_\Delta\Delta - \epsilon + \dot{r}_\Delta\gamma)\lambda_0 + (\kappa - \dot{r}_\Delta\tau)\lambda_1 = 0 \quad (51)$$

where we use $\gamma = 0, \pi = \overline{\tau} \hat{=} 0, \kappa = O(r')$. The total six equations that include two time-related conditions and four spatial-related conditions are

$$
\begin{aligned}
\dot{\lambda}_0^0 - \epsilon_0\lambda_0^0 &= 0, \text{i.e.,} \quad \text{þ}\lambda_0^0 = \dot{r}_\Delta \text{þ}'\lambda_0^0 = 0, && (a)\\
\dot{\lambda}_1^0 + \epsilon_0\lambda_1^0 &= 0, \text{i.e.,} \quad \text{þ}\lambda_1^0 = \dot{r}_\Delta \text{þ}'\lambda_1^0 = 0, && (b)\\
\eth_0\lambda_0^0 + \sigma_0\lambda_1^0 &= 0 \;\; \text{from } \eth\lambda_0 + \sigma\lambda_1 = 0, && (c)\\
\eth_0\lambda_1^0 - \mu_0\lambda_0^0 &= 0 \;\; \text{from } \eth\lambda_1 - \mu\lambda_0 = 0, && (d)\\
\eth_0'\lambda_0^0 &= 0 \;\; \text{from } \overline{\eth}\lambda_0 + \rho\lambda_1 = 0, && (e)\\
\eth_0'\lambda_1^0 - \lambda_0\lambda_0^0 &= 0 \;\; \text{from } \overline{\eth}\lambda_1 + \lambda\lambda_0 = 0, && (f)
\end{aligned}
$$

From the commutation relations, we have $\text{þ}_0\eth_0 - \eth_0\text{þ}_0 \hat{=} \sigma_0\eth_0'$. From (a), we have $\eth_0\text{þ}_0\lambda_0^0 = 0$. From (c) and use the fact $\text{þ}_0\sigma_0 = 0$, we have $\text{þ}_0\eth_0\lambda_0^0 = -\sigma_0\text{þ}_0\lambda_1^0$. To make this relation compatible with commutation relation, we need an extra condition

$$\text{þ}_0\lambda_1^0 = -\overline{\eth}_0\lambda_0^0. \quad (g)$$

To examine whether this extra condition is compatible with Dougan-Mason holomorphic condition, we apply þ_0 on (d). From (d), we have

$$\text{þ}_0\eth_0\lambda_1^0 = (\sigma_0\lambda_0 + \Psi_2^0)\lambda_0^0. \quad (52)$$

From (g) and together use the fact $\overline{\eth}_0\sigma_0 = 0$ and commutation relation of $[\eth, \eth']$, we have

$$\eth_0\text{þ}_0\lambda_1^0 = \sigma_0\eth_0'\lambda_1^0 + (\sigma_0\lambda_0 + \Psi_2^0)\lambda_0^0. \quad (53)$$

Therefore, this extra condition is compatible with Dougan-Mason holomorphic condition.

From the previous analysis, we concludes that the compatible frame alignment conditions for the non-rotating dynamical horizon would be one time related condition (54) (i.e., (a)), two Dougan-Mason holomorphic conditions (55) (i.e., (c)) and (56) (i.e., (d)) and one extra condition (57) (i.e., (g)):

$$\begin{aligned} þ_0\lambda_0^0 &= \dot{r}_\Delta þ'\lambda_0^0 = 0, \text{ i.e., } \dot{\lambda}_0^0 - \epsilon_0\lambda_0^0 = 0, && (54)\\ ð_0\lambda_0^0 + \sigma_0\lambda_1^0 &= 0, && (55)\\ ð_0\lambda_1^0 - \mu_0\lambda_0^0 &= 0, && (56)\\ þ_0\lambda_1^0 &= -\bar{ð}_0\lambda_0^0, && (57) \end{aligned}$$

where first three conditions are similar with the compatible constant spinor conditions for the generic isolated horizon. If the shear term vanishes, we do not need this extra condition.

Remark. The Dougan-Mason holomorphic conditions (55) and (56) will tell us how to gauge fix the horizon quasi-local energy expression and the time related condition (54) will tell us how the energy momentum change with time along the dynamical horizon.

V. ENERGY-MOMENTUM AND FLUX

A. The quasi-local energy-momentum of an isolated horizon

By using Nester-Witten two form together with the compatible constant spinor conditions which are Dougan-Mason's holomorphic conditions (39) and (40) for the NEH, the quasi-local momentum integral near a NEH is

$$\begin{aligned} I(r') &= -\tfrac{1}{8\pi}\oint_{S_r}[\lambda_{0'}ð\lambda_1 - \lambda_1ð\lambda_{0'} + \lambda_0ð'\lambda_{1'} - \lambda_{1'}ð'\lambda_0 \\ &\quad -\lambda_0\lambda_{0'}(\mu+\overline{\mu}) - \lambda_1\lambda_{1'}(\rho+\overline{\rho})]dS \\ &= \tfrac{1}{4\pi}\oint_S[-\mu_0\lambda_0^0\overline{\lambda}_{0'}^0 + O(r')]dS. \end{aligned} \tag{58}$$

Moreover, the horizon momentum $P_{\underline{AA'}}$ can be written as

$$P_{\underline{AA'}}(S_\Delta) = I(r_\Delta)\lambda_{\underline{A}}\overline{\lambda}_{\underline{A'}} \tag{59}$$

where $\lambda_{\underline{A}}$ is constant spinor on two surface of NEH. From the result of the asymptotic expansion for the generic isolated horizons, we can re-interpret the *quasi-local energy-momentum integral of the generic isolated horizons (NEH)* as

$$I(r_\Delta) = -\frac{1}{4\pi}\oint_S \frac{\Psi_2^0 - \dot{\mu}_0 + ð_0\pi_0 + \pi_0\overline{\pi}_0}{2\epsilon_0}\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \tag{60}$$

where $\Psi_2^0 = M + iL$ and $ð_0\pi_0 = A - iL$ with M, L, A are function of (v,θ,ϕ). We compare with Kerr solution in the Appendix A.

B. News function of the generic isolated horizons

In order to match the Kerr solution that its flux vanishes, we rescale the spinor field. Firstly, the constant spinors λ_0^0 and λ_1^0 are rescaled by using the following relation

$$\tilde{\lambda}_0^0 = \lambda_0^0 e^{-\int\epsilon_0 dv}, \quad \tilde{\lambda}_1^0 = \lambda_1^0 e^{-\int\epsilon_0 dv}, \tag{61}$$

and it yields the new rescaled momentum integral

$$\tilde{I}(r_\Delta) = e^{-2\int\epsilon_0 dv}I(r_\Delta) = -\frac{1}{4\pi}\oint \mu_0\tilde{\lambda}_0^0\tilde{\overline{\lambda}}_{0'}^0 dS_\Delta. \tag{62}$$

The three compatible conditions (37), (39) and (40) then become

$$\dot{\tilde{\lambda}}_0^0 = 0, \quad ð_0\tilde{\lambda}_0^0 = 0, \quad ð_0\tilde{\lambda}_1^0 - \mu_0\tilde{\lambda}_0^0 = 0 \tag{63}$$

where we use $\delta_0\epsilon_0 = 0$ from asymptotic expansion and they are still compatible under rescaling.

By using this new rescaling constant spinor frame, we apply the time derivative on the quasi-local energy-momentum of NEH (58) and thus we get

$$\dot{\tilde{I}}(r_\Delta) = -\frac{1}{4\pi}\oint \dot{\mu}_0\tilde{\lambda}_0^0\tilde{\overline{\lambda}}_{0'}^0 dS_\Delta \tag{64}$$

We said that (64) is *quasi-local energy flux near NEH*. Here $\dot{\mu}_0$ is related with the mass loss or gain, hence it is the *news function of NEH*. However, since the black hole laws do not hold on NEH, we conclude that *the news function of NEH is not physically reasonable. This is due to that one cannot measure the correct temperature of a black hole since he or she uses a bad thermometer, which corresponds to the normalization of ℓ. If one choose the canonical $[\ell]$ which yield WIH, then $\dot{\mu}_0$ vanishes.*

C. Conserved quantities of the generic isolated horizons

We now consider the absolute conservation law that $\dot{G}_m = 0$ on the generic isolated horizons. From **(b)** in p. 49, we have $\dot{\Psi}_2^0 = 0$, therefore, we can find ten conserve quantities which are

$$G_m = \int {}_2Y_{2,m}\Psi_2^0 dS. \quad (m = -2, -1...., 2) \tag{65}$$

Here these conserved quantities corresponds to three different type of generic IHs. These conserved quantities are related with mass and angular momentum of the generic IHs.

D. Quasi-local energy momentum of a non-rotating dynamical horizon

By using the Nester-Witten two form and Dougan-Mason's holomorphic conditions (55) and (56) which we found in pervious section, the quasi-local energy-momentum integral on horizon can be expressed as

$$I(r_\Delta) = -\frac{1}{4\pi}\oint \mu_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. \tag{66}$$

Then use **(h)** in p. 46, the quasi-local energy-momentum integral for a non-rotating dynamical horizon can be expressed as

$$\begin{aligned} I(r_\Delta) &= -\frac{1}{4\pi}\oint \frac{1}{2\epsilon_0}[\Psi_2^0 - \dot{\mu}_0 - \sigma_0{\sigma'}_0 && (67)\\ &\quad +\dot{r}_\Delta(\mu_0^2 + {\sigma'}_0{\overline{\sigma}'}_0)]\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. && (68)\end{aligned}$$

E. Flux expression of a non-rotating dynamical horizon

Analysis and properties of NP equations

<1> From **(a)** in p. 46, because of dominate energy condition, we have $\dot{r}_\Delta(\Psi_2^0 - \sigma_0{\sigma'}_0) = -\Phi_{00}^0 - \sigma_0\overline{\sigma}_0 \leq 0$ and $\dot{r}_\Delta \geq 0$. It implies $\Psi_2^0 - \sigma_0{\sigma'}_0 \leq 0$.

<2> From imaginary of **(l)** in p. 46, it implies $Im\Psi_2^0 = (\sigma_0{\sigma'}_0 - \overline{\sigma}_0{\overline{\sigma}'}_0)$. If $\mathrm{Im}\Psi_2^0 = 0$ for shear non-zero, then it implies ${\sigma'}_0 = -A\overline{\sigma}_0$ where A is real and time independent.

<3> Use **<2>** and from the result of asymptotic expansion **(b)** in p. 46 and **(g)** in p. 46, we get $0 = (-4A\epsilon_0 + A\dot{r}_\Delta\mu_0 + \mu_0)\sigma_0 + \dot{r}_\Delta\overline{\Psi}_4^0$. (1) If $\dot{r}_\Delta = 0$, it implies $(-4A\epsilon_0 + \mu_0)\sigma_0 = 0$. So either $\sigma_0 = 0$ or $\mu_0 = 4A\epsilon_0$. For the later case $\dot{r}_\Delta = 0$ does not imply shear vanishes. Since $\dot{r}_\Delta$ should imply shear vanishing, this case does not satisfy the boundary conditions of NEHs. (2) If $\sigma_0 = 0$, then there are two situation. (a) $\dot{r}_\Delta = 0$ then it goes back to NEH. (b) $\Psi_4^0 = 0$ but $\dot{r}_\Delta$ not vanish. It is similar to Vaidya.

<4> From **(g)** in p. 46, if ${\sigma'}_0 = 0$, i.e., $A = 0$, then $\mu_0\overline{\sigma}_0 = -\dot{r}_\Delta\Psi_4^0$. (1) If $\dot{r}_\Delta = 0$ since μ_0 cannot be zero then $\sigma_0 = 0$. (2) If $\sigma_0 = 0$, then there are two situation. (2') $\dot{r}_\Delta = 0$ then it goes back to NEH. (2") $\Psi_4^0 = 0$ but $\dot{r}_\Delta$ not vanish. It is similar to Vaidya.

From **<3>** and **<4>**, it would be more reasonable for us to chose ${\sigma'}_0 = 0$, i.e., $A = 0$.

<5> If we choose the coordinate $r_\Delta = -\frac{1}{\mu_0}$, then μ_0 is not related with θ or ϕ. So $\overline{\eth}_0\mu_0 = 0$. Together with the choice ${\sigma'}_0 = 0$, we then have $\Psi_3^0 = 0$ from **(m)** in p. 47.

From the above analysis and reasons, we choose $r_\Delta = -\frac{1}{\mu_0}$ and set

$${\sigma'}_0 = 0$$

to derive flux expression. From **(g)** in p. 46, **(b)** in p. 49 and **(h)** in p. 46, we have

$$\begin{aligned} \mu_0\overline{\sigma}_0 &= -\dot{r}_\Delta\Psi_4^0, && (69)\\ \dot{\Psi}_2^0 &= \mu_0\Phi_{00}^0 + 6\mu_0^2\epsilon_0\dot{r}_\Delta + \mu_0\sigma_0\overline{\sigma}_0, && (70)\\ \Psi_2^0 &= 2\epsilon_0\mu_0. && (71)\end{aligned}$$

The energy-momentum integral (68) then becomes

$$\begin{aligned} I(r_\Delta) &= -\frac{1}{4\pi}\oint \mu_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \quad \text{(i)}\\ &= -\frac{1}{4\pi}\oint \frac{\Psi_2^0}{2\epsilon_0}\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \quad \text{(ii)}\end{aligned}$$

Now we apply time derivative on the energy-momentum integral and also we use the time related conditions (54) of spinor field.

Apply time derivative on (i), we get

$$\dot{I}(r_\Delta) = \frac{1}{4\pi}\oint \dot{\mu}_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. \qquad (72)$$

Apply time derivative on (ii), we get

$$\dot{I}(r_\Delta) = -\frac{1}{4\pi}\int \frac{\mu_0}{2\epsilon_0}[\sigma_0\overline{\sigma}_0 + \Phi_{00}^0]\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. \qquad (73)$$

Finally, we get the quasi-local energy-momentum flux formula which is obtained from asymptotic expansion for the non-rotating non-spherical symmetric dynamical horizon (shear non-vanishing on the boundary) is

$$\begin{aligned} \dot{I}(r_\Delta) &= \frac{1}{4\pi}\oint \dot{\mu}_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta\\ &= -\frac{1}{4\pi}\int \frac{\mu_0}{2\epsilon_0}[\sigma_0\overline{\sigma}_0 + \Phi_{00}^0]\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \qquad (74)\end{aligned}$$

which is quite similar with Ashtekar's flux equation for dynamical horizon.

Therefore, $\dot{\mu}_0$ is *the news function of non-rotating DHs* that determines gravitational radiation and matter field radiation. The shear square term $|\sigma_0|^2$ is believed to be related with gravitational radiation, since the gravitational radiation comes from the non-spherical symmetric gravitational collapse. In the later section, we will see more detail about how this equation can be compared with Ashtekar's flux formula.

F. The relationship between Ashtekar-Krishnan flux and our flux formula

From (i)

From the choice of $\mu_0 = -\frac{1}{r_\Delta}$, we have $\dot{\mu}_0 = \frac{\dot{r}_\Delta}{r_\Delta^2} = \frac{\dot{r}_\Delta}{2}\,{}^{(2)}R$ where the two scalar curvature is ${}^{(2)}R = \frac{2}{r_\Delta^2}$ (The metric of a two sphere with radius r_Δ is $dl^2 = -r_\Delta^2(d\theta^2 + \sin^2\theta d\phi^2)$.). We use these relations and substitute them into the the time derivative of (i) (recall (72)) in previous section, we then obtain

$$\begin{aligned} \dot{I}(r_\Delta) = \frac{dI}{dv} &= \frac{1}{4\pi}\oint \dot{\mu}_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta\\ &= \frac{1}{8\pi}\oint \frac{dr_\Delta}{dv}\,{}^{(2)}R\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta.\end{aligned}$$

Integrate the above equation with respect to v, we then have

$$dI = \frac{1}{8\pi} \int {}^{(2)}R\lambda_0^0 \overline{\lambda}_{0'}^0 dS_\Delta dr_\Delta. \quad (75)$$

We recall that the Ashtekar's total flux formula [2] is

$$F_{matter} + F_{grav} = \frac{1}{16\pi} \int_{\Delta H} {}^{(2)}RNd^3V \quad (76)$$

which $F_{matter} + F_{grav}$ is equal to flux dI and $d^3V = dr_\Delta dS$ on horizon. Therefore, if $N = 2\lambda_0^0 \overline{\lambda}_{0'}^0$, the our flux formula from equation (i) is completely the same with Ashtekar-Krishnan's formula (76).

From (ii)

Recall the Ashtekar-Krishnan gravitational flux expression (14) is

$$F_{grav} = \frac{1}{4\pi} \int_{\Delta H} N(|\sigma|^2 + |\pi|^2) d^3V \quad (77)$$

where it satisfies the gauge condition of our asymptotic expansion. The matter flux expression of Vaidya solution is

$$F_{\text{matter}} := \int_H T_{ab} T^a \ell^b N d^3V \quad (78)$$

$$= \frac{1}{4\pi} \int \Phi_{00} N d^3V \quad (79)$$

where we use $4\pi T_{ab}\ell^a\ell^b = \Phi_{00}^0$. The total Ashtekar-Krishnan flux of non-rotating DH [2] [10] then becomes

$$F_{\text{total}} = \frac{1}{4\pi} \int [|\sigma|^2 + \Phi_{00}] N d^3V. \quad (80)$$

Recall our formula (73) in previous section and then integrate it with respect to v, we have

$$\begin{aligned} dI(r_\Delta) &= -\frac{1}{4\pi} \int \frac{\mu_0}{2\epsilon_0} [\sigma_0 \overline{\sigma}_0 + \Phi_{00}^0] \lambda_0^0 \overline{\lambda}_{0'}^0 dS_\Delta dv \\ &= -\frac{1}{4\pi} \int \frac{\mu_0}{2\epsilon_0 \dot{r}_\Delta} [\sigma_0 \overline{\sigma}_0 + \Phi_{00}^0] \lambda_0^0 \overline{\lambda}_{0'}^0 dS_\Delta dr_\Delta \end{aligned}$$

where $dv = \frac{dr_\Delta}{\dot{r}_\Delta}$ (Since dr' vanishes on horizon.).

1. Using **(h)** in p. 46 and **(l)** in p. 46, we have

$$2\epsilon_0\mu_0 = -P \overset{c}{\nabla} \alpha_0 - \overline{P} \overset{\bar{c}}{\nabla} \overline{\alpha}_0 + 4\alpha_0\overline{\alpha}_0 = -\frac{1}{2r_\Delta^2} = -\frac{1}{2}\mu_0^2$$

   where we use the fact that for a sphere metric $\alpha_0 = -\frac{\sqrt{2}\cot\theta}{4r_\Delta}$. It then implies

$$\mu_0 = -4\epsilon_0. \quad (81)$$

2. Because v is arbitrary, one can always rescale v so that we can chose $\dot{r}_\Delta = 1$.

From 1. and 2. and with the choice of $N = 2\lambda_0^0 \overline{\lambda}_{0'}^0 > 0$, we then have $-\frac{\mu_0}{2\epsilon_0 \dot{r}_\Delta} = 2$. Our formula is then the same as Ashtekar-Krishnan's formula (80). Therefore, the surface gravity $\kappa_{(\ell)}$ is $\kappa_{(\ell)} = \frac{1}{2r_\Delta}$. The differential of the horizon area is $dA = 8\pi r_\Delta dr_\Delta$ and $\kappa_{(\ell)} dA = 4\pi dr_\Delta$. For the time evolute vector $t^a = N\ell^a$, the difference of the horizon energy E^t can be expressed as

$$\int dE^t = \int dI(r_\Delta) = \frac{1}{4\pi} \int [|\sigma|^2 + \Phi_{00}] N d^3V \quad (82)$$

$$= \int \frac{1}{2} dr_\Delta = \int \frac{\kappa_{(\ell)}}{8\pi} dA \quad (83)$$

which it is always positive. We can get *a generalized black hole first law for non-rotating DHs*

$$\frac{\kappa_{(\ell)}}{8\pi} dA = dE^t. \quad (84)$$

VI. CONCLUSIONS

We know that the news function determines the gravitational radiation near the null infinity. It is the time derivative of the shear of outgoing null tetrad ($\dot{\overline{\sigma}}_0$). Similarly, the horizon news function can determine the gravitational radiation and energy loss or gain near the horizon. Our work of the asymptotic expansions near quasi-local horizons does not require the spacetime is asymptotically flat. Using the method of asymptotic expansions we find that the corresponding news functions near quasi-local horizons are the time derivative of the expansion of the incoming null normal ($\dot{\mu}_0$). If we apply the time derivative on the Bondi mass, it will give us a minus quadratic of time derivative of the shear term. It is always *negative*! Therefore, for an isolated gravitating system, it will always lost mass and the gravitational wave will carry gravitational radiation out to infinity.

The news function of dynamical horizon determines the quasi-local energy flux cross the dynamical horizon to be always *positive*, if the dominate energy condition holds. Then the dynamical horizon will gain mass and the horizon will grow. In this paper, we have investigated the quasi-local energy of quasi-local horizons for which the physical energy conditions hold.

Asymptotically constant spinors can be used to define the quasi-local energy-momentum of the horizon. Searching for the compatible conditions of constant spinors of horizons offers us a way to chose for the proper reference frames when measuring these quasi-local quantities. In practical, Dougan-Mason's holomorphic conditions refer to how to fix the gauge (good measurement) in quasi-local energy-momentum expression (Nester-Witten two form) on each cross section of horizon. The time related condition can tell us how the quasi-local energy-momentum change with time along horizon.

We find that the news function exists for the NEHs. It indicates two possibilities: one is that there are gravitational radiations outside the NEHs, and the other is that gravitational radiations cross to the equilibrium black hole but the horizon area

will not increase. For the first one, we *cannot* prove whether news function corresponding to gravitational radiations outside the NEHs. For the second one, it would be against black hole thermodynamic laws. However, since the news function of the NEHs can be made to be vanished while we make a special choice of affine parameter $r_\Delta = -\frac{1}{\mu_0}$, we conclude that *this is due to a bad measurement (bad gauge choice) of gravitational flux*. This result refers to that a generic NEH admits a unique $[\ell]$ such that $(\Delta, [\ell])$ is a WIH on which the incoming expansion μ_0 is time independent [1].

Laws of black hole dynamics

Zeroth law For NEHs or DHs, the zeroth law will not hold. If we make a coordinate choice $r_\Delta = -1/\mu_0$ for NEH, then WIH will preserve the zeroth law mechanics.

First law From our construction, we can derive the generalized first law eq. (84) for non-rotating DHs.

Second law It is obvious that our flux formula for a dynamical horizon is related with the black hole area law. If the news function $\dot{\mu}_0$ is positive, i.e., the positive energy flux. From the fact of the area of cross section changing with time $\frac{d}{dv}dS_\Delta = 2r_\Delta\dot{\mu}_0 dS_\Delta$, we know that the black hole area is always increasing if the dominate energy condition holds. Therefore, the black hole second law can extend to a dynamical horizon.

For a spherical symmetric dynamical horizon, the contribution of energy radiation is purely from the matter field. Therefore the gravitational contribution is from the non-spherical symmetric term: the shear term. Our result about the gravitational radiation of non-rotating DHs that comes from the next order contribution of asymptotic expansion is proportional to the shear square and always positive. This result agrees with perturbation method of energy flux cross event horizon [Done by Hawking and Hartle, see Chandrasekhar [5]] and Ashtekar-Krishnan's flux formula for non-rotating DH.

One can refer to Chapter 6 of [9] or ref [10] for the relations between energy fluxes and area balance of rotating dynamical horizons. The future work on gravitational radiation and angular momentum flux of rotating DHs is in progress [10].

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APPENDIX A: QUASI-LOCAL MASS FOR THE KERR SOLUTION

Here we present three different calculations for Kerr horizon mass. In Subsection A 2, we re-calculate Bergqvist's calculation for Kerr horizon mass [3]. He used the Dougan-Mason mass definition and solve the holomorphic conditions to get the exact expressions for the constant spinors. To calculate the Dougan-Mason mass on the bounded two sphere, i.e., at the quasi-local level, we need a tetrad to satisfy the gauge conditions (ρ, μ are real). Therefore, we need to make a null rotation of Kerr solution in advance Eddington-Frankelstein coordinate. By using the constant spinors, the new NP coefficients and the Dougan-Mason mass, we can calculate the quasi-local mass for Kerr. In order to compare with our work, we expand it with repect to angular parameter a when a is small.

To make our tetrad to be compatible with the method of the asymptotic expansion for Kerr, we transfer it to the Bondi coordinate. Thus we have the desire gauge conditions in Bondi coordinate when a is small in Kerr metric. By using the results of the Kerr solution in the approximate Bondi frame in Subsection A 1, we can solve the holomorphic conditions and calculate the quasi-local mass by using Dougan-Mason's definition in Subsection A 3.

In Subsection A 4, we check the Kerr solution by using our quasi-local mass that obtain from asymptotic expansion (60) and see whether it can compare with the other two resuls in the previous sections. Our conclusion is that all these three calculations are the same up to the second order when power expanding with respect to the angular momentum parameter a.

1. Kerr solution in the approximate Bondi frame

We transfer Kerr solution from advanced Eddinton-Frankelstein cooordinates to Bondi coordinate near horizon (detail see Chapter 6 of [9]). The Kerr metric in Bondi coordinate $(\tilde{v}, \tilde{r'}, \tilde{\theta}, \tilde{\phi})$ near the horizon is therefore given by

$$
\begin{aligned}
g^{00} &= O(r'^2), \ \ g^{03} = O(r'^2), \ \ g^{12} = O(r'^2), \ \ g^{02} = O(r'^2),\\
g^{01} &= -\frac{{r_\Delta}^2 + a^2}{\Sigma_\Delta} + [\frac{2r_\Delta a^2 \sin^2\theta}{\Sigma_\Delta^2} + \frac{a^2\sin^2\theta(r_\Delta - M)}{(r_\Delta^2 + a^2)\Sigma_\Delta}]r'\\
&+ O(r'^2),\\
g^{13} &= -\frac{a}{\Sigma_\Delta} + [\frac{a(r_\Delta - M)(2{r_\Delta}^2 + 2a^2 - a^2\sin^2\theta)}{({r_\Delta}^2 + a^2)^2\Sigma_\Delta}\\
&+ \frac{2ar_\Delta}{{\Sigma_\Delta}^2}]r' + O(r'^2),\\
g^{22} &= -\frac{1}{\Sigma_\Delta} + 2\frac{r_\Delta}{{\Sigma_\Delta}^2}r' + O(r'^2),\\
g^{33} &= -\frac{1}{\sin^2\theta\Sigma_\Delta} + \frac{a^2(2{r_\Delta}^2 + 2a^2 - a^2\sin^2\theta)}{2({r_\Delta}^2 + a^2)^2\Sigma_\Delta} + O(r'),\\
g^{23} &= -\frac{2a^3\sin\theta\cos\theta}{({r_\Delta}^2 + a^2)^2\Sigma_\Delta}r' + O(r'^2),\\
g^{11} &= -2\frac{r_\Delta - M}{\Sigma_\Delta}r' + O(r'^2),
\end{aligned}
$$

where $r_\Delta := r_+ = M + \sqrt{M^2 - a^2}$ and $\Sigma_\Delta := r_\Delta^2 + a^2\cos^2\theta$.

Now we consider the case of slow rotation so that a is small, the tetrad components in the Bondi coordinate $(\tilde{v}, \tilde{r'}, \tilde{\theta}, \tilde{\phi})$ are:

$$\ell^a = (1, Ur', 0, \frac{a}{r_\Delta{}^2} + Dr'), \quad \text{(A1)}$$

$$n^a = (0, -1, 0, 0), \quad \text{(A2)}$$

$$m^a = \frac{1}{\sqrt{2}\eta_\Delta}(0, 0, 1 - \frac{r'}{\eta_\Delta}, \frac{-i}{\sin\theta}(1 - \frac{r'}{\eta_\Delta})), \quad \text{(A3)}$$

where $\eta_\Delta := r_\Delta + ia\cos\theta$, $U := \frac{r_\Delta - M}{r_\Delta^2}$ and $D := \frac{a(2r_\Delta - M)}{r_\Delta^4}$. The NP coefficients and Weyl tensors are:

$$\begin{aligned}
\kappa &= \sigma = \lambda = \nu \hat{=} 0,\\
\rho &= \frac{U(-r_\Delta + r')r'}{(\eta_\Delta - r')(\overline{\eta}_\Delta - r')} \hat{=} 0,\\
\mu &= \frac{-r_\Delta + r'}{(\eta_\Delta - r')(\overline{\eta}_\Delta - r')} \hat{=} -\frac{r_\Delta}{\Sigma_\Delta},\\
\pi &= \overline{\tau} = \frac{i\sqrt{2}D\eta_\Delta^2\sin\theta}{4(\eta_\Delta - r')} \hat{=} \frac{i\sqrt{2}D\eta_\Delta\sin\theta}{4},\\
\beta &\hat{=} -\frac{\sqrt{2}}{8}iD\sin\theta\overline{\eta}_\Delta + \frac{\sqrt{2}r_\Delta\cot\theta}{4\Sigma_\Delta}\\
&\quad -\frac{\sqrt{2}ia\cos(2\theta)}{4\sin\theta\Sigma_\Delta}, \qquad \text{(A4)}\\
\epsilon &= \frac{U[(r' - r_\Delta)^2 + a^2\cos^2\theta + ia\cos\theta r']}{2[(r' - r_\Delta)^2 + a^2\cos^2\theta]} \hat{=} \frac{U}{2},\\
\gamma &\hat{=} -\frac{ia\cos\theta}{2\Sigma_\Delta}, \gamma + \overline{\gamma} \hat{=} 0,\\
\pi &\hat{=} \alpha + \overline{\beta},\\
\Psi_0 &\hat{=} 0, \Psi_1 \hat{=} 0,\\
\mathrm{Im}\Psi_2 &\hat{=} -\frac{iD\cos\theta}{\Sigma_\Delta}(r_\Delta^2 + a^2\cos^2\theta - a^2\sin^2\theta),\\
\Psi_3 &\hat{=} \frac{i\sqrt{2}\sin\theta r_\Delta\eta_\Delta}{4\Sigma_\Delta^3}[D\Sigma_\Delta^2 + 2ia^2\cos^2\theta],\\
\Psi_4 &\hat{=} 0.
\end{aligned}$$

From this Kerr tetrad in the approximate Bondi frame, the NP coefficients satisfy

$$\begin{aligned}
&\nu \hat{=} \mu - \overline{\mu} \hat{=} \pi - \alpha - \overline{\beta} \hat{=} \gamma + \overline{\gamma} \hat{=} \epsilon - \overline{\epsilon} \hat{=} 0,\\
&\pi \hat{=} \overline{\tau}, \rho \hat{=} \overline{\rho}, \mu < 0. \qquad \text{(A5)}
\end{aligned}$$

2. Bergqvist's calculation

After making the null rotation, the complex spatial null tetrad is

$$m^a = \frac{1}{\sqrt{2}\overline{\eta}_\Delta}[\partial_\theta - \frac{i\Sigma_\Delta}{\sin\theta(r_\Delta^2 + a^2)}\partial_\phi], \quad \text{(A6)}$$

and the NP β then becomes [3]

$$\beta(\theta) = \frac{1}{2\sqrt{2}\overline{\eta}_\Delta}[\cot\theta - \frac{2ia\sin\theta}{\overline{\eta}_\Delta} - \frac{ia(r - M)\sin\theta}{(r_\Delta^2 + a^2)}].$$

The Dougan-Mason's holomorphic conditions can be written as two equations in component. The one is

$$0 = \eth\lambda_0 = (\delta - \beta)\lambda_0, \quad \text{(A7)}$$

where the shear vanishes. The solution has the form $\lambda_0 = C_{-1}A_{-1}(\theta)e^{-i\phi/2} + C_1A_1(\theta)e^{i\phi/2}$. The A_{-1}, A_1 satisfy the differential equations

$$\partial_\theta A_{-1} - (G + H)A_{-1} = 0, \quad \text{(A8)}$$

$$\partial_\theta A_1 + (G - H)A_1 = 0. \quad \text{(A9)}$$

With the aid of using Maple, we can find the solutions are

$$\begin{aligned}
A_{-1}(\theta) &= e^{\int G + H d\theta} = \tfrac{\sin(\theta/2)}{\overline{\eta}_\Delta}\exp(\tfrac{ia\cos\theta(r - M - ia)}{2(r_\Delta^2 + a^2)}),\\
A_1(\theta) &= e^{\int -G + H d\theta} = \tfrac{\cos(\theta/2)}{\overline{\eta}_\Delta}\exp(\tfrac{ia\cos\theta(r - M + ia)}{2(r_\Delta^2 + a^2)}),
\end{aligned}$$

where

$$G := \frac{\Sigma_\Delta}{2\sin\theta(r_\Delta^2 + a^2)}, \quad H := \sqrt{2}\overline{\eta}_\Delta\beta. \quad \text{(A10)}$$

The other holomorphic condition is

$$0 = \eth\lambda_1 - \mu\lambda_0 = (\delta + \beta)\lambda_1 - \mu\lambda_0. \quad \text{(A11)}$$

The solution has the form $\lambda_1 = B_{-1}(\theta)e^{-i\phi/2} + B_1(\theta)e^{i\phi/2}$. It satisfies the differential equations

$$\partial_\theta B_{-1} - (G - H)B_{-1} - \sqrt{2}\mu\overline{\eta}_\Delta C_{-1}A_{-1} = 0, \quad \text{(A12)}$$

$$\partial_\theta B_1 + (G + H)B_1 - \sqrt{2}\mu\overline{\eta}_\Delta C_1A_1 = 0, \quad \text{(A13)}$$

where

$$B_1(\theta) = \frac{C_1}{\sqrt{2}A_{-1}}\int h(s)ds, \quad \text{(A14)}$$

$$B_{-1}(\theta) = \frac{C_{-1}}{\sqrt{2}A_1}\int h(s)ds, \quad \text{(A15)}$$

and

$$\int h(\theta)d\theta = \int 2\mu\eta_\Delta A_1A_{-1}d\theta. \quad \text{(A16)}$$

Use $\mu = -\frac{r_\Delta}{r_\Delta^2 + a^2} - \frac{(r_\Delta - M)a^2\sin^2\theta}{2(r_\Delta^2 + a^2)^2}$, we get

$$\begin{aligned}
&\textstyle\int_0^\pi \mu|A_{-1}|^2\sin\theta d\theta\\
&= -\tfrac{\tan^{-1}(\frac{a}{r})(r^2 + a^2)(r + M) - ar(M - r)}{2ar(r^2 + a^2)^2}, \qquad \text{(A17)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\pi h(s)ds = \int_0^\pi 2\mu\eta_\Delta A_1A_{-1}d\theta\\
&= \frac{1}{4}[\sqrt{r^2 + a^2}a(2M - 2r + 1) - 3r^3\ln(\frac{a + \sqrt{r^2 + a^2}}{-a + \sqrt{r^2 + a^2}})\\
&+2(r + M)a^2\ln(-a + \sqrt{r^2 + a^2})\\
&-Mr^2\ln\ln(\frac{a + \sqrt{r^2 + a^2}}{-a + \sqrt{r^2 + a^2}}) - 2a^2M\ln a\\
&-2ra^2\ln(a + \sqrt{r^2 + a^2})]/(a(r^2 + a^2)^2). \qquad \text{(A18)}
\end{aligned}$$

Therefore, we use $-(A17)/|A18|$ to calculate Dougan-Mason mass.Thus we get

$$\begin{aligned} m_{DM} &= 2Mr_+ \frac{\int_0^\pi -\mu(\theta)|A_{-1}(\theta)|^2 \sin\theta d\theta}{|\int_0^\pi h(\theta)d\theta|} \\ &= M - \frac{a^2}{24M} + O(a^4). \end{aligned} \quad \text{(A19)}$$

3. Using the Dougan-Mason conditions to calculate the Kerr mass in the approximate Bondi frame

The complex spatial null tetrad for Kerr in Bondi coordinate has the form

$$m^a = \frac{1}{\sqrt{2}\eta_\Delta}(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi). \quad \text{(A20)}$$

The NP β is function of θ on horizon (recall eq. (A4)) and it has the form

$$\beta(\theta) = -\frac{\sqrt{2}}{8} iD \sin\theta \overline{\eta}_\Delta + \frac{\sqrt{2}r_\Delta \cot\theta}{4\Sigma_\Delta} - \frac{\sqrt{2}ia\cos(2\theta)}{4\sin\theta\Sigma_\Delta}.$$

Dougan-Mason's holomorphic conditions in components has two equations. One is

$$0 = \eth\lambda_0 = (\delta - \beta)\lambda_0. \quad \text{(A21)}$$

The solution has the form $\lambda_0 = C_{-1}A_{-1}(\theta)e^{-i\phi/2} + C_1 A_1(\theta)e^{i\phi/2}$ where A_1, A_{-1} satisfy

$$\partial_\theta A_{-1} - (G+H)A_{-1} = 0, \ \ \partial_\theta A_1 + (G-H)A_1 = 0.$$

We use Maple to check the solutions are

$$\begin{aligned} A_{-1}(\theta) &= e^{\int G+H d\theta} \\ &= \sin(\theta/2)\sqrt{\overline{\eta}_\Delta}\exp(i\cos\theta D(3r_\Delta^2 + a^2\cos^2\theta)/12), \\ A_1(\theta) &= e^{\int -G+H d\theta} \\ &= \cos(\theta/2)\sqrt{\overline{\eta}_\Delta}\exp(i\cos\theta D(3r_\Delta^2 + a^2\cos^2\theta)/12), \end{aligned}$$

$$G := \frac{1}{2\sin\theta}, \quad H := \sqrt{2}\eta_\Delta\beta. \quad \text{(A22)}$$

The other is

$$0 = \eth\lambda_1 - \mu\lambda_0 = (\delta + \beta)\lambda_1 - \mu\lambda_0. \quad \text{(A23)}$$

The solution has the form $\lambda_1 = B_{-1}(\theta)e^{-i\phi/2} + B_1(\theta)e^{i\phi/2}$. B_1, B_{-1} satisfy

$$\partial_\theta B_{-1} - (G-H)B_{-1} - \sqrt{2}\mu\eta_\Delta C_{-1}A_{-1} = 0, \quad \text{(A24)}$$

$$\partial_\theta B_1 + (G+H)B_1 - \sqrt{2}\mu\eta_\Delta C_1 A_1 = 0, \quad \text{(A25)}$$

where

$$B_1(\theta) = \frac{C_1}{\sqrt{2}A_{-1}}\int h(s)ds, \quad \text{(A26)}$$

$$B_{-1}(\theta) = \frac{C_{-1}}{\sqrt{2}A_1}\int h(s)ds, \quad \text{(A27)}$$

and

$$\int h(\theta)d\theta = \int 2\mu\eta_\Delta A_1 A_{-1} d\theta. \quad \text{(A28)}$$

For Kerr solution in Bondi coordinate, we have $\mu = -\frac{r_\Delta}{\Sigma_\Delta}$. Thus we get

$$\begin{aligned} m_{DM} &= 2Mr_+ \frac{\int_0^\pi -\mu(\theta)|A_{-1}(\theta)|^2 \sin\theta d\theta}{|\int_0^\pi h(\theta)d\theta|} \\ &= M - \frac{a^2}{24M} + O(a^4) \end{aligned} \quad \text{(A29)}$$

which is approximate to the Kerr mass when a is very small.

4. Checking our quasi-local formula

We replace μ_0 by using $\frac{\Psi_2^0 + \eth_0\pi_0 + \pi_0\overline{\pi}_0}{2\epsilon_0}$ in order to check our quasi-local formula. Since we know the NP coefficients and Weyl tensors for Kerr in Bondi coordinate (See Section A 1), we have

$$\frac{\Psi_2^0 + \eth_0\pi_0 + \pi_0\overline{\pi}_0}{2\epsilon_0} = -\frac{r_\Delta}{3\Sigma_\Delta} - \frac{\sin^2\theta\Sigma_\Delta D^2}{24U} - \frac{E}{24U\Sigma_\Delta^3},$$

where

$$E := 8\Sigma_\Delta^2 - 4r_\Delta^4 + 4a^2(a^2\cos^2\theta - r_\Delta^2\sin^2\theta). \quad \text{(A30)}$$

We thus get

$$\begin{aligned} m_{DM} &= 2Mr_+ \frac{\int_0^\pi -[\frac{\Psi_2^0 + \eth_0\pi_0 + \pi_0\overline{\pi}_0}{2\epsilon_0}]|A_{-1}(\theta)|^2 \sin\theta d\theta}{|\int_0^\pi h(\theta)d\theta|} \\ &= M - \frac{a^2}{24M} + O(a^4), \end{aligned} \quad \text{(A31)}$$

which is approximate to the Kerr mass when angular momentum per mass is very small.

Remark. We conclude that the quasi-local mass for these three cases are the same up to a^2. This is not surprising because they are based on Dougan-Mason's quasi-local mass definition. Even though our work for the Kerr in Bondi coordinate is just approximation, we still can get the same quasi-local mass up to the second order. However, if one compare with the Hawking mass, the second order is different from the Dougan-Mason mass. Hawking mass of Kerr horizon is

$$\begin{aligned} m_H &= M[\frac{1 + (1 - a^2/M^2)^{\frac{1}{2}}}{2}]^{\frac{1}{2}} \\ &= M - \frac{a^2}{8M} + O(a^4). \end{aligned} \quad \text{(A32)}$$

APPENDIX B: QUASI-LOCAL MASS AND ENERGY FLUX FOR THE VAIDYA SOLUTION

A good example to study the dynamical horizon is Vaidya solution. For this, we know that the dynamical horizon is

a space-like hypersurface if the black hole is growing. On the contrary, if black hole is contracting then it's a time-like hypersurface. In the later case the dominate energy condition will not hold so we restrict attention to the former case. The outgoing null normal of the Vaidya solution is $\ell^a = \frac{\partial}{\partial v} + (\frac{r'}{2(r'+r_\Delta)} - \dot{r}_\Delta)\frac{\partial}{\partial r'}$, we also have $\rho\hat{=}\sigma\hat{=}0$ on horizon, and $\Phi_{00} = \frac{\dot{M}}{r^2}$. We now show that if the dominate energy condition is satisfied then $\dot{r}_\Delta > 0$. From the Raychaudhuri equation $\mathcal{L}_\ell\Theta_{(\ell)} = -\Theta^2_{(\ell)} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 2\epsilon\Theta_{(\ell)} + \frac{1}{2}R_{ab}\ell^a\ell^b$. The expansion of the outgoing null normal of Vaidya is $\Theta_{(\ell)} = -Re\rho = \frac{r-2M(v)}{2r^2}\hat{=}\frac{r'}{2r^2_\Delta}$, so it vanishes on horizon. However, it's time derivative does not vanish. Here we have $\mathcal{L}_\ell\Theta_{(\ell)}\hat{=} -\frac{\dot{r}_\Delta}{2r^2_\Delta}\hat{=} -\Phi_{00} \leq 0$. If the dominate energy condition satisfied, then $\Phi_{00} \geq 0$, i.e., $\dot{r}_\Delta \geq 0$. Let V denote a vector field which is tangential to the dynamical horizon H^+, is everywhere orthogonal to the foliation by marginally trapped surfaces and preserves this foliation. We can always choose the normalization of ℓ^a and n^a such that $\ell^a n_a = 1$ and $V^a = \ell^a - fn^a$ for some f. Since $V^aV_a = -2f$, it follows that H^+ is space-like, null or time-like, depending on whether f is positive, zero or negative. We now show that if the dominate energy condition holds then f is non-negative. Let us begin by noting that the definition of V^a immediately implies $\mathcal{L}_V\Theta_{(\ell)} = 0$, whence, $\mathcal{L}_\ell\Theta_{(\ell)} = f\mathcal{L}_n\Theta_{(\ell)}$. Therefore, the Raychaudhuri equation for ℓ^a implies $f\mathcal{L}_n\Theta_{(\ell)} = -|\sigma|^2 - \Phi_{00} \leq 0$. Physically, we can assume that the expansion of the outgoing null normal becomes negative if it moves along the incoming direction to the interior of the marginally trapping surfaces, so $\mathcal{L}_n\Theta_{(\ell)} < 0$. Hence, if the dominate energy condition is satisfied, then f is non-negative, i.e., the dynamical horizon H^+ is space-like or null. The Vaidya solution is similar with the Schwarzschild solution but the mass is time dependent and it's a non-vacuum dust solution. In this appendix we calculate our quasi-local mass and flux for the Vaidya solution and compare with the results one expects from the form of Vaidya metric. In advanced Eddington-Finkelstein coordinates (v, r, θ, ϕ), the metric is

$$ds^2 = (1 - \tfrac{2M(v)}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{B1}$$

where the horizon radius $r_\Delta(v) = 2M(v)$ is time dependent. When r approaches the horizon radius $r_\Delta(v)$, the metric approaches to

$$ds^2\hat{=} - 2\dot{r}_\Delta dv^2 - r^2_\Delta(d\theta^2 + \sin^2\theta d\phi^2) \tag{B2}$$

where we use a coordinate transformation $dr' = dr - \dot{r}_\Delta dv$. It's obvious that the Vaidya horizon is space-like or null hypersurface since it has the metric signature $(---)$ if $\dot{r}_\Delta \geq 0$. It's null when the mass approach to constant and it returns to the Schwarzschild solution. In the new coordinate (v, r', θ, ϕ), the tetrad components are

$$\begin{array}{lllll} \ell^a &= (1 ,& \frac{r'}{2(r'+r_\Delta)} - \dot{r}_\Delta ,& 0 ,& 0) \\ n^a &= (0 ,& -1 ,& 0 ,& 0) \\ m^a &= (0 ,& 0 ,& \frac{1}{\sqrt{2}(r'+r_\Delta)} ,& -\frac{i}{\sqrt{2}(r'+r_\Delta)\sin\theta}) \end{array} \tag{B3}$$

The NP coefficients are

$$\begin{aligned} &\kappa = \sigma = \lambda = \nu = \tau = \pi = \gamma = 0, \\ &\rho = -\tfrac{r-2M(v)}{2r^2} = \tfrac{r'}{2r^2_\Delta} - \tfrac{r'^2}{r^3_\Delta} + O(r'^3), \\ &\mu = -\tfrac{1}{r} = -\tfrac{1}{r_\Delta} + \tfrac{r'}{r^2_\Delta} + O(r'^2), \\ &\epsilon = \tfrac{M(v)}{2r^2} = \tfrac{1}{4r_\Delta} + O(r'), \\ &\alpha = -\beta = -\tfrac{\sqrt{2}\cot\theta}{4r_\Delta} + O(r'), \\ &\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2 = -\tfrac{M(v)}{r^3_\Delta} + O(r'), \\ &\Phi_{00} = \tfrac{\dot{M}(v)}{r^2_\Delta} + O(r'). \end{aligned} \tag{B4}$$

Quasi-local mass of Vaidya horizon

Let's calculate the horizon energy momentum near the Vaidya horizon and examine the related mass gain or loss formula near such dynamical horizon. We use the holomorphic gauge fixing, so the horizon quasi-local energy-momentum is

$$P^{\underline{AA'}} = -\frac{1}{4\pi}\oint_S \mu\lambda^{\underline{A}}_0\lambda^{\underline{A'}}_{0'}dS. \tag{B5}$$

Here we use the Dougan-Mason definition of mass:

$$m^2_{DM} = P_{\underline{AA'}}P^{\underline{AA'}} = \frac{2}{|N|^2}(P^{\underline{00'}}P^{\underline{11'}} - P^{\underline{01'}}P^{\underline{10'}}) \tag{B6}$$

where $N = \varepsilon^{\underline{01}} = \lambda^{\underline{0}}_0\lambda^{\underline{1}}_1 - \lambda^{\underline{0}}_1\lambda^{\underline{1}}_0$. To calculate the Dougan-Mason mass, we have to solve the holomorphic equations for the constant spinors equations. Here we simply follow the calculation that done by Bergvist [3]. The holomorphic conditions:

$$\eth\lambda_0 + \sigma\lambda_1 = 0 \Rightarrow \qquad (\delta - \beta)\lambda_0 = 0 \tag{B7}$$

$$\eth\lambda_1 - \mu\lambda_0 = 0 \Rightarrow \ (\delta + \beta)\lambda_1 - \mu\lambda_0 = 0 \tag{B8}$$

For $\eth\lambda_0 = 0$, we have to solve the PDE

$$(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cot\theta}{2})\lambda_0 = 0, \tag{B9}$$

then we get the solution $\lambda_0 = C_{-1}\lambda^{\underline{0}}_0 + C_1\lambda^{\underline{1}}_0$. For $\eth\lambda_1 - \mu\lambda_0 = 0$, we have to solve the PDE

$$\frac{1}{\sqrt{2}r_\Delta}[(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cot\theta}{2})\lambda_1 + \sqrt{2}\lambda_0] = 0, \tag{B10}$$

then we get $\lambda_1 = C_{-1}\lambda^{\underline{0}}_1 + C_1\lambda^{\underline{1}}_1$. Here

$$\begin{array}{lll} \lambda^{\underline{0}}_0 &=& A_{-1}e^{-i\phi/2}, \ \lambda^{\underline{1}}_0 = A_1e^{i\phi/2}, \\ \lambda^{\underline{0}}_1 &=& \frac{\int^\theta_\pi B(s)ds\ e^{-i\phi/2}}{\sqrt{2}A_1(\theta)}, \ \lambda^{\underline{1}}_1 = \frac{\int^\theta_0 B(s)ds\ e^{i\phi/2}}{\sqrt{2}A_{-1}(\theta)}, \\ A_1 &=& \cos(\theta/2)/r, \ A_{-1} = \sin(\theta/2)/r, \\ B(\theta) &=& \sin\theta/r^2. \end{array} \tag{B11}$$

We choose two independent solution $C_{-1} = 1, C_1 = 0$ for $\lambda^{\underline{0}}_A$ and $C_{-1} = 0, C_1 = 1$ for $\lambda^{\underline{1}}_A$ here. So when we integrate over ϕ, $\phi \leq \phi \leq 2\pi$ in $P^{\underline{AA'}}$, we find $P^{\underline{01'}} = P^{\underline{10'}}$ and $P^{\underline{00'}} = P^{\underline{11'}}$. Therefore, the Dougan-Mason mass becomes

$$m_{DM} = \frac{\sqrt{2}P^{\underline{00'}}}{|N|} \tag{B12}$$

where

$$P^{00'} = -\frac{1}{4\pi}\int \mu|A_{-1}|^2 dS^2 \hat{=} \frac{1}{2r_\Delta} \tag{B13}$$

and $N = \lambda_0^0\lambda_1^1 - \lambda_1^0\lambda_0^1 = \frac{1}{\sqrt{2}}\int_0^\pi h(\theta)d\theta = -\frac{\sqrt{2}}{r^2}$. Finally, the Vaidya horizon mass from the Dougan-Mason's mass definition is

$$m_{DM} = \frac{r_\Delta}{2} = M(v) \tag{B14}$$

which agrees with the expected result.

Quasi-local flux of Vaidya horizon

Next, we reduce our flux formula to Vaidya by taking shear vanishing and compare it by using the solution of the holomorphic constant spinors. For Vaidya solution, we have

$$\mu_0 = -\frac{1}{r_\Delta(v)}, \sigma_0' = \sigma_0 = 0 \tag{B15}$$

where r_Δ is time dependent. We substitute these formulae for the NP quantities from the Vaidya solution into our quasi-local energy-momentum integral we obtain in (68). Then

$$I(r_\Delta) = -\frac{1}{4\pi}\oint \mu_0\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \tag{B16}$$

$$= -\frac{1}{4\pi}\oint \frac{\Psi_2^0}{2\epsilon_0}\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. \tag{B17}$$

Then we rescale λ_0^0 so that we get the condition $\dot{\lambda}_0^0 = 0$ (recall (54)) from our asymptotic constant spinor condition for the dynamical horizon. We use the fact that $\frac{\partial}{\partial v}dS_\Delta = 2\frac{\dot{r}_\Delta}{r_\Delta}dS_\Delta$.

From the first expression (B16), the quasi-local energy flux is

$$\frac{dI}{dv} = -\frac{1}{4\pi}\oint(\dot{\mu}_0 + 2\mu_0\frac{\dot{r}_\Delta}{r_\Delta})\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta \tag{B18}$$

$$= \frac{1}{4\pi}\oint \frac{\dot{M}}{2M^2}\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta. \tag{B19}$$

From the second expression (B17), it becomes

$$\frac{dI}{dv} = -\frac{1}{4\pi}\oint(\frac{\dot{\Psi}_2^0}{2\epsilon_0} - \frac{\Psi_2^0\dot{\epsilon}_0}{2\epsilon_0^2} + \frac{\Psi_2^0\dot{r}_\Delta}{\epsilon_0 r_\Delta})\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta$$

$$(\text{Use } \Psi_2^0 = -\frac{1}{8M^2}, \epsilon_0 = \frac{1}{8M})$$

$$= \frac{1}{4\pi}\oint \frac{\dot{M}}{2M^2}\lambda_0^0\overline{\lambda}_{0'}^0 dS_\Delta.$$

We use the Dougan-Mason's mass definition, then $\dot{m}_{DM} = \frac{\sqrt{2}\dot{P}^{00'}}{|N|}$. Therefore we can put $\lambda_0^0 = \lambda_0^{\underline{0}}$ in our quasi-local formula. The Vaidya's quasi-local mass flux is

$$\frac{dm_{DM}}{dv} = \frac{r^2}{4\pi}\int \frac{\dot{M}}{2M^2}\frac{\sin(\theta/2)^2}{r^2}dS = \dot{M}. \tag{B20}$$

If the dominate energy condition satisfied, i.e., $\dot{M} \geq 0.$, then we find the mass gain law for the Vaidya.

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